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A BEST APPROXIMATION THEOREM FOR NONEXPANSIVE SET-VALUED MAPPINGS IN HYPERCONVEX METRIC SPACES

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1. Introduction. Recent results have shown that many fixed point and best approximation theorems previously established for Banach spaces have analogues in hyperconvex metric spaces, see, for example, [**2, 4–6**]. In [**4**] the authors gave a hyperconvex version of the Fan best approximation theorem for set-valued mappings on compact sets. It is the purpose of this paper to show that a best approximation theorem can be obtained in hyperconvex spaces for set-valued mappings without compactness assumptions, under the additional requirement that the mappings are nonexpansive. This result is applied to obtain some fixed point theorems.

2. Preliminaries. Using $B(x,r)$ to denote the closed ball with center $x \in M$ and radius r, a metric space (M,d) is hyperconvex if, given any family $\{x_\alpha\}$ of points in M and any family $\{r_\alpha\}$ of real numbers satisfying $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$, it is the case that $\cap B(x_\alpha, r_\alpha) \neq \varnothing$. Hyperconvex metric spaces were introduced and their basic properties elaborated in [**1**].

The *externally hyperconvex* subsets (relative to M), denoted by $E(M)$, are those subsets S such that, given any family $\{x_\alpha\}$ of points in M and any family $\{r_\alpha\}$ of real numbers satisfying $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ and $d(x_\alpha, S) \leq r_\alpha$, it follows that $S \cap (\cap B(x_\alpha, r_\alpha)) \neq \varnothing$. Throughout, $d(x,S) = \inf_{y \in S} d(x,y)$ for any subset S.

The admissible subsets of M, denoted by $A(M)$, are sets of the form $\cap B(x_\alpha, r_\alpha)$, i.e., the family of ball intersections in M. Admissible subsets are externally hyperconvex [**1**]. A subset S is *proximinal* provided for each $x \in M$ there is an $s \in S$ such that $d(x,s) = d(x,S)$.

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