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THE HENSTOCK VARIATIONAL MEASURE, BAIRE FUNCTIONS AND A PROBLEM OF HENSTOCK

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1. Introduction. It is well known that the Lebesgue integral is a proper extension of the Riemann integral. Henstock [**9**] and Kurzweil [**11**] independently gave a slight, but ingenious, modification of the classical Riemann integral to obtain a Riemann-type definition of the Perron integral, which is an extension of the Lebesgue integral. This relatively new integral is now commonly known as the Henstock-Kurzweil integral [**1**–**3, 12, 15**], Kurzweil-Henstock integral [**10, 14**], the gauge integral [**18**] or the Henstock integral [**8, 13**]. In this paper, we shall use the term "Henstock-Kurzweil integral" for this integral.

The original definition of the Henstock-Kurzweil integral, see Definition 2.2, involves completely arbitrary positive gauge function δ. Bullen in [**4**] raised the question of determining how complicated δ need be. It turns out that a measurable positive gauge function can be selected for the one-dimensional Henstock-Kurzweil integral; see for example [**7, 8, 12, 13**]. For the importance of this result in topology, see [**12**]. Foran and Meinershagen went further to prove that if F is generalized absolutely continuous in the restricted sense on a compact interval $[a, b]$ in $\mathbf{R}$ with

$$f(x) = \begin{cases} F'(x) & \text{if } F'(x) \text{ exists,} \\ 0 & \text{otherwise,} \end{cases}$$

then the positive gauge function δ in the definition of the Henstock-Kurzweil integral of f can be chosen to be Baire 2 everywhere. See [**7**, Theorem 2] for more details. Since their method of proof is real-line dependent, Henstock in [**10**, pp. 53–54] asked whether an analogous result holds in higher dimensions. In this paper, we give an affirmative answer to the above problem of Henstock. Moreover, we deduce a full descriptive characterization of the Henstock-Kurzweil integral [**15**, Theorem 4.3]. It is worthwhile to note that, unlike the method

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