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INTEGRAL-GEOMETRIC FORMULAS FOR PERIMETER IN $\mathbf{S}^2$, $\mathbf{H}^2$ AND HILBERT PLANES

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Dedicated to the memory of our friend and colleague Felix Albrecht.

ABSTRACT. We develop two types of integral formulas for the perimeter of a convex body K in planar geometries. We derive Cauchy-type formulas for perimeter in planar Hilbert geometries. Specializing to $\mathbf{H}^2$ we get a formula that appears to be new. In the projective model of $\mathbf{H}^2$ we have $\mathcal{P} = (1/2)\int w\,d\phi$. Here w is the Euclidean length of the projection of K from the ideal boundary point $R = (\cos\phi, \sin\phi)$ onto the diametric line perpendicular to the radial line to R (the image of K may contain points outside the model). We show that the standard Cauchy formula $\mathcal{P} = \int \sinh r\,d\omega$ in $\mathbf{H}^2$ follows, where ω is a central angle perpendicular to a support line and r is the distance to the support line.

The Minkowski formula $\mathcal{P} = \int \kappa_g \rho^2\,d\theta$ in $\mathbf{E}^2$ generalizes to $\mathcal{P} = 1/(4\pi^2)\int \kappa_g L(\rho)^2\,d\theta + k/2\pi \int A(\rho)\,ds$ in $\mathbf{H}^2$ and $\mathbf{S}^2$. Here (ρ, θ) and κ_g are, respectively, the polar coordinates and geodesic curvature of ∂K, k is the (constant) curvature of the plane, and $L(\rho)$ and $A(\rho)$ are, respectively, the perimeter and area of the disk of radius ρ. In $\mathbf{E}^2$ this is locally equivalent to the Cauchy formula $\mathcal{P} = \int r\,d\omega$ in the sense that the integrands are pointwise equal. In contrast, the corresponding Minkowski and Cauchy formulas are not locally equivalent in $\mathbf{H}^2$ and $\mathbf{S}^2$.

1. Introduction.

1.1 Overview. There are at least two natural integral-geometric approaches relating the perimeter of a convex body K in $\mathbf{E}^2$ to its other geometric properties. There is the beautiful Cauchy formula

$$\mathcal{P} = \frac{1}{2}\int_0^{2\pi} w\,d\phi, \tag{1.1}$$

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