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AN ISOPERIMETRIC INEQUALITY FOR RIESZ CAPACITIES

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ABSTRACT. Let A be a compact set of $\mathbf{R}^n$, and let A^* be the closed ball centered at the origin with the same measure as A. We prove that, if C_α is the α-Riesz capacity with $0 < \alpha < 2$, then $C_\alpha(A) \geq C_\alpha(A^*)$. We also prove an isoperimetric inequality for the expected measure of the stable sausage generated by A. Our results also yield isoperimetric inequalities for the relativistic α-stable processes, and other Lévy processes.

1. Introduction. It is well known that, among all compact sets of equal measure, the ball has the smallest Newtonian capacity. This is one of the classical generalized isoperimetric inequalities of Pólya and Szegő [**11**]. In [**8**], Luttinger provided a new method, based on multiple integrals inequalities, to prove this and many other isoperimetric inequalities. In this paper we adapt the method of Luttinger [**8**] to obtain isoperimetric inequalities for Riesz capacities. The Riesz kernel is

$$k_\alpha(x-y) = \frac{\Gamma(n-\alpha/2)}{\Gamma(\alpha/2)\,\pi^{n/2}2^{\alpha-1}}\;\frac{1}{|x-y|^{n-\alpha}},$$

where $n \geq 2$ and $0 < \alpha < n$. Let A be a compact set in $\mathbf{R}^n$, the α-Riesz capacity of A is defined by

$$C_\alpha(A) = \Big[\inf_\mu \iint k_\alpha(x-y)\; d\mu(x)\, d\mu(y)\Big]^{-1},$$

where the infimum is taken over all probability Borel measures supported in A. If $\alpha = 2$ and $n \geq 3$, then this is the classic Newtonian capacity. Let $|A|$ be the Lebesgue measure of A, and let A^* be the

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