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Φ-INEQUALITIES OF NONCOMMUTATIVE MARTINGALES

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ABSTRACT. In the recent article [**10, 11**], Pisier and Xu showed that, among other things, the noncommutative analogue of the classical Burkholder-Gandy inequalities in martingale theory. We prove the noncommutative analogue of the classical Φ-inequalities for commutative martingale.

1. Preliminaries. Let E be a rearrangement invariant space on $[0,\infty)$, cf. [**5**] for the definition. We denote by $\mathcal{N}$ a semi-finite von Neumann algebra with a semi-finite normal faithful trace σ. The set of all $\sigma-$measurable operators will be denoted by $\widetilde{\mathcal{N}}$. For $x \in \widetilde{\mathcal{N}}$, let $\mu_{\cdot}(x)$ be the generalized singular value function of x, cf. [**4**]. We define

$$L_E(\mathcal{N},\sigma) = \big\{x \in \widetilde{\mathcal{N}} : \ \mu_{\cdot}(x) \in E\big\}$$
$$\|x\|_{L_E(\mathcal{N},\sigma)} = \|\mu_{\cdot}(x)\|_E \quad \text{for} \quad x \in L_E(\mathcal{N},\sigma).$$

Then $(L_E(\mathcal{N},\sigma), \|.\|_{L_E(\mathcal{N},\sigma)})$ is a Banach space, [**2, 12**]. For $E = L^p(0,\infty)$, we recover the noncommutative L^p-space $L^p(\mathcal{N},\sigma)$ associated with $(\mathcal{N},\sigma)$. We will denote $L_E(\mathcal{N},\sigma)$ simply by $L_E(\mathcal{N})$. Let $a = (a_n)_{n\geq 0}$ be a finite sequence in $L_E(\mathcal{N})$, define

$$\|a\|_{L_E(\mathcal{N},l^2_C)} = \Big\| \Big(\sum_{n\geq 0} |a_n|^2 \Big)^{1/2} \Big\|_{L_E(\mathcal{N})},$$
$$\|a\|_{L_E(\mathcal{N},l^2_R)} = \Big\| \Big(\sum_{n\geq 0} |a_n^*|^2 \Big)^{1/2} \Big\|_{L_E(\mathcal{N})}.$$

This gives two noms on the family of all finite sequences in $L_E(\mathcal{N})$. To see this, denoting by $\mathcal{B}(l^2)$ the algebra of all bounded operators on l^2 with its usual trace tr, let us consider the von Neumann algebra tensor

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