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A SIMPLER FUBINI PROOF

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ABSTRACT. We give a new and simpler proof of the Fubini theorem. The proof uses a different definition of measurability which allows for a more geometric approach than usual.

1. Introduction. The definition of the product outer measure is straightforward. The measure of a set in the product is the infimum of outer approximation by sums of rectangular areas. The difficulties, and they are substantial, arise in the proof of the "obvious" relationship between the integral with respect to the product measure and the iterated integrals with respect to the measures on the two spaces. The basic problem is to show that if $\lambda = \mu \times \nu$, and E is λ-measurable, then

$$\lambda(E) = \iint \chi_E(x, y)\, d\nu(y)\, d\mu(x). \tag{1}$$

Equation (1) is the simplest case of the Fubini theorem, and also its essential core, for the general result follows easily from the special case (1).

The difficulty in verifying (1) lies in showing that the sections of a λ-measurable set E are measurable with respect to μ and ν, and that $\int \chi_E(x, y)\, d\nu(y)$ is a measurable function of x. If there is a suitable topology available, as for instance in $[0, 1] \times [0, 1]$, then compactness can be used to simplify matters. In the general (non-topological) case, the standard arguments all involve a skein of set theory which effectively hides the geometry. See, e.g., [**1**, pp. 135–147], [**2**, pp. 143–148], [**3**, pp. 303–310], [**4**, pp. 147–151].

We give here a proof for a general product which gives a clearer picture of how close approximation of a product set by rectangles forces a close approximation to sections by measurable sets in the factor spaces. The proof depends on a formally weaker condition for measurability than the usual Carathéodory condition. This condition

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