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ON QUADRATIC SOLUTIONS OF $x^4 + py^4 = z^4$

ERIC D. MANLEY

ABSTRACT. Consider the diophantine equation $x^4+py^4 = z^4$ where p is prime and $p \equiv 3 \bmod 8$. It is well known that this equation has no nonzero integer solutions. This paper shows that all quadratic solutions are inherited. That is, all quadratic solutions can be easily obtained from integer solutions to the simpler equation $x^4 + py^4 = z^2$.

1. Introduction. One of Fermat's most well-known results is the nonexistence of nonzero integer solutions to $x^4 + y^4 = z^4$. In 1934, Aigner proved that nonzero *quadratic* solutions exist though they are rare [**1**]. In fact, $\mathbf{Q}(\sqrt{-7})$ is the only quadratic extension with nonzero solutions. Observe, $(1 + \sqrt{-7})^4 + (1 - \sqrt{-7})^4 = 2^4$. Faddeev later classified all solutions in $\mathbf{Q}(\sqrt{-7})$ [**3**]. The complexity of Faddeev's methods motivated Mordell to supply an alternative argument [**5**].

We are interested in generalizing Aigner's results to the family of equations $x^4 + Dy^4 = z^4$ with $D \in \mathbf{Z}$. We prove that the only quadratic solutions to $x^4 + py^4 = z^4$ with $p \equiv 3 \bmod 8$ are those that come from rational solutions of $x^4 + py^4 = z^2$. For example, since $(1)^4 + 3(1)^4 = (2)^2$, we find $(1, 1, \sqrt{2}) \in \mathbf{Q}(\sqrt{2})^3$ satisfies $x^4 + 3y^4 = z^4$. That is, we will prove

Theorem 1. *All quadratic solutions to $x^4+py^4 = z^4$ for $p \equiv 3 \bmod 8$ can be written in the form $(a, b, \sqrt{c})$ where (a, b, c) is a rational solution to $x^4 + py^4 = z^2$.*

Furthermore, we will show

Corollary 2. *All quadratic solutions to $x^4 + py^4 = z^4$ with $p \equiv 11 \bmod 16$ satisfy $xyz = 0$.*

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