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# MONOTONICITY PROPERTIES AND INEQUALITIES OF FUNCTIONS RELATED TO MEANS

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ABSTRACT. In this paper, monotonicity properties of functions related to means are discussed and some inequalities are established.

**1. Introduction.** The generalized logarithmic mean (Stolarsky mean) $L_r(a,b)$ of two positive numbers $a,b$ is defined in [**1, 2**] for $a=b$ by $L_r(a,b)=a$ and for $a\neq b$ by

$$\begin{aligned} L_r(a,b) &\triangleq \left(\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\right)^{1/r}, \quad r\neq -1,\ 0;\\ L_{-1}(a,b) &= \frac{b-a}{\ln b-\ln a}\triangleq L(a,b);\\ L_0(a,b) &= \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}\triangleq I(a,b), \end{aligned}$$

when $a\neq b$, $L_r(a,b)$ is a strictly increasing function of $r$. Clearly,

$$L_1(a,b)\triangleq A(a,b), \quad L_{-2}(a,b)\triangleq G(a,b),$$

where $A$ and $G$ are the arithmetic and geometric means, respectively.

The logarithmic mean $L(a,b)$ is generalized to the one-parameter mean in [**3**]:

$$\begin{aligned} J_r(a,b) &\triangleq \frac{r(b^{r+1}-a^{r+1})}{(r+1)(b^r-a^r)}, \quad a\neq b, \quad r\neq 0,-1;\\ J_0(a,b) &\triangleq L(a,b);\\ J_{-1}(a,b) &\triangleq \frac{[G(a,b)]^2}{L(a,b)};\\ J_r(a,a) &\triangleq a, \end{aligned}$$

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