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THE HURWITZ ZETA FUNCTION AS A CONVERGENT SERIES

ROMAN DWILEWICZ AND JÁN MINÁČ

ABSTRACT. New series for the Hurwitz zeta function which converge on the whole plane, except $s = 1$, are developed. This is applied to obtain a remarkably simple evaluation of some special values of the function.

1. Introduction. Classically the Riemann zeta function, or more generally, the Hurwitz zeta function, is defined on a half plane using a series and then it is analytically extended, with respect to s, to the whole plane except for a simple pole at $s = 1$ with residue 1,

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \quad \text{for} \quad \Re s > 1 \quad \text{and} \quad 0 < x \leq 1,$$

however in many calculations x can be taken any positive number. The Riemann zeta function is obtained from the Hurwitz function by setting $x = 1$. In this paper we define the Hurwitz zeta function by a series which converges on the whole plane except for $s = 1$. In fact we define a family of series parameterized by certain easily constructible sequences of natural numbers $\{g_n\}_{n=0}^{\infty}$. Our constructions and proofs are elementary and they require only the basic properties of Bernoulli numbers (for basic properties of Bernoulli numbers and L-functions we refer the reader to [**3**] and [**23**]) and complex analysis of one variable, see, e.g., [**46**]. The new series leads to a very simple and natural evaluation of L-functions at negative integers. One example of our series is the following:

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