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THE WEAK CHANG-MARSHALL INEQUALITY VIA GREEN'S FORMULA

MIROSLAV PAVLOVIĆ AND DRAGAN VUKOTIĆ

ABSTRACT. We prove the uniform Trudinger-Moser type inequality of Chang and Marshall for the Dirichlet space when $\alpha < 1$ by using only Green's formula instead of Beurling's deep inequalities.

1. Introduction. In this note we present a very short proof of the weak Chang-Marshall inequality based only on Green's formula for the disk and a standard growth estimate for the functions in the Dirichlet space $\mathcal{D}$ of the disk. By the *weak Chang-Marshall inequality* we mean the uniform estimate

$$(1)\qquad \sup\left\{\int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2}\,d\theta : \|f\|_{\mathcal{D}} \le 1,\ f(0)=0\right\} < \infty, \quad \alpha < 1.$$

This is a complex variable case of the well-known inequalities of Trudinger-Moser type. The uniform estimate (1) no longer holds when $\alpha > 1$. Its proof in the critical case $\alpha = 1$ was a deep result of Chang and Marshall [**3**] and provided an answer to a question stated on page 1079 of Moser's influential paper [**6**]. See also [**5**] for a simplified proof and [**2**] for more details and the vast literature on this topic and its relations with geometry.

The weak Chang-Marshall inequality is certainly easier to prove than the case $\alpha = 1$. However, its proofs that one encounters in the literature are based on the following deep uniform estimate from Beurling's thesis [**1**]:

$$(2)\qquad \text{If } f\in\mathcal{D},\ \|f\|_{\mathcal{D}}\le 1, \quad\text{and}\quad f(0)=0, \quad\text{then}\quad |E_\lambda| \le e^{-\lambda^2+1}.$$

Here $E_\lambda = \{\theta \in [0,2\pi] : |f(e^{i\theta})| > \lambda\}$ and $|E_\lambda|$ is its normalized arc measure on the unit circle $\mathbf{T}$. Namely, a generalization of the basic

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