ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 5, 2006

A SUMMATION FORMULA FOR SEQUENCES INVOLVING FLOOR AND CEILING FUNCTIONS

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ABSTRACT. A closed form expression for the Nth partial sum of the pth powers of $\|\sqrt{n}\|$ is obtained, where $\|\cdot\|$ denotes the nearest integer function. As a consequence, a necessary and sufficient condition for the divisibility of n by $\|\sqrt{n}\|$ is derived together with a closed form expression for the least nonnegative residue of n modulo $\|\sqrt{n}\|$. In addition an identity involving the zeta function $\xi(s)$ and the infinite series $\sum_{n=1}^{\infty} 1/\|\sqrt{n}\|^{s+1}$ for real $s>1$ is also obtained.

1. Introduction. In a recent paper, see [**3**], the author examined the problem of determining a closed form expression for those sequences $\langle b_m \rangle$ formed from an arbitrary sequence of real numbers $\langle a_n \rangle$ in the following manner. Let $d \in \mathbf{N}$ be fixed, and for each $m \in \mathbf{N}$ define b_m to be the mth term of the sequence consisting of nd occurrences in succession of the terms a_n, as follows:

$$\underbrace{a_1, \dots, a_1}_{d,\ a_1 \text{ terms}}, \underbrace{a_2, \dots, a_2}_{2d,\ a_2 \text{ terms}}, \underbrace{a_3, \dots, a_3}_{3d,\ a_3 \text{ terms}}, \dots . \tag{1}$$

For example, if $a_n = n$ and $d = 1$ then the resulting sequence $\langle b_m \rangle$ would be

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \dots .$$

Specifically, the problem described above required the construction of a function $f : \mathbf{N} \to \mathbf{N}$ such that $b_m = a_{f(m)}$. As was shown in [**3**] the required function $f(\cdot)$ can easily be described in terms of a combination of floor and ceiling functions, that is the functions defined as $\lfloor x \rfloor = \max\{n \in \mathbf{Z} : n \le x\}$ and $\lceil x \rceil = \min\{n \in \mathbf{Z} : x \le n\}$ respectively. In particular, for the sequence in (1), we have that $b_m = a_{f(m)}$ where

$$f(m) = \left\lfloor \sqrt{\left\lceil \frac{2m}{d} \right\rceil} + \frac{1}{2} \right\rfloor . \tag{2}$$

Received by the editors on November 12, 2003.

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