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THE COMPLETE CONTINUITY PROPERTY IN BANACH SPACES

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ABSTRACT. Let X be a complex Banach space. We show that the following are equivalent: (i) X has the complete continuity property, (ii) for every, or equivalently for some, $1 < p < \infty$, for $f \in h^p(\mathbf{D}, X)$ and $r_n \uparrow 1$, the sequence f_{r_n} is p-Pettis-Cauchy, where f_{r_n} is defined by $f_{r_n}(t) = f(r_n e^{it})$ for $t \in [0, 2\pi]$, (iii) for every, or equivalently for some, $1 < p < \infty$, for every $\mu \in V^p(X)$, the bounded linear operator $T : L^q(0, 2\pi) \to X$ defined by $T\phi = \int_0^{2\pi} \phi \, d\mu$ is compact, where $1/q + 1/p = 1$, (iv) for every, or equivalently for some, $1 < p < \infty$, each $\mu \in V^p(X)$ has a relatively compact range.

Before stating our results we overview the involved concepts and notations of vector-valued harmonic analysis. Throughout this note $(X, \|\cdot\|)$ denotes a complex Banach space, $\mathbf{D}$ denotes the open unit disc in the complex plane, and λ is the normalized Lebesgue measure on $[0, 2\pi]$. For a Banach space Y, we denote by B_Y the closed unit ball of Y. Given $1 < p < \infty$ the space $h^p(\mathbf{D}, X)$ consists of all X-valued harmonic functions f on $\mathbf{D}$ such that

$$\|f\|_p = \sup_{0<r<1} \left(\int_0^{2\pi} \|f(re^{it})\|^p \, d\lambda(t) \right)^{1/p} < \infty.$$

Accordingly, $h^\infty(\mathbf{D}, X)$ is the space of all X-valued bounded harmonic functions on $\mathbf{D}$ equipped with the norm $\|f\|_\infty = \sup_{z \in \mathbf{D}} \|f(z)\|$. For $f \in h^p(\mathbf{D}, X)$ and $n \in \mathbf{Z}$, the Fourier coefficient $\hat{f}(n)$ is computed as

$$\hat{f}(n) = r^{-|n|} \int_0^{2\pi} f(re^{it}) e^{-int} \, d\lambda(t).$$

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