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THE STORY OF A TOPOLOGICAL GAME

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ABSTRACT. In the author's dissertation, he introduced a simple topological game. Seemingly minor variations of this game have over the years seen various uses, including the characterization of Corson and Eberlein compacta, and characterizing when certain function spaces with the compact-open topology are Baire. This article is primarily a survey of this game and its applications. Some new results are included, and a number of open problems are stated.

1. Introduction. Let X be a topological space, and $x \in X$. The following four games will be discussed in this paper:

(1) $G_{O,P}(X,x)$: In the nth round, Player O chooses an open neighborhood O_n of x, and Player P chooses a point $p_n \in \cap_{i \leq n} O_i$. O *wins* if the sequence $\{p_n\}_{n\in\omega}$ converges to x.

(2) $G_{K,P}(X)$: In the nth round, Player K chooses a compact subset K_n of X, and P chooses a point $p_n \notin \cup_{i\leq n} K_i$. K *wins* if the sequence $\{p_n\}_{n\in\omega}$ is a closed discrete subset of X.

(3) $G_{K,L}(X)$: In the nth round, K chooses a compact subset K_n of X, and L chooses a compact subset L_n of X disjoint from $\cup_{i\leq n} K_i$. K *wins* if $\{L_n\}_{n\in\omega}$ is a closed discrete collection in X.

(4) $G^o_{K,L}(X)$: Same as (3), except that K *wins* if $\{L_n\}_{n\in\omega}$ has a discrete open expansion.

These four games are variations on the same theme. In fact, note that if X is compact, then $G_{O,P}(X,x)$ is equivalent to the game $G_{K,P}(X\backslash\{x\})$. Of course, $G_{K,L}(X)$ is essentially the game $G_{K,P}(X)$ modified to allow P to choose compact sets instead of single points.

$G_{O,P}(X,x)$ was introduced in [**18**], where it was helpful in solving a problem of Zenor, and studied in detail in [**19**], where it was used to define and study a new convergence property. $G_{K,P}(X)$ was, in effect, introduced in [**20**], where it was used to characterize Corson compact spaces and strong Eberlein compact spaces, as well as the

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