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INEQUALITIES OF GENERALIZED HYPERBOLIC METRICS

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ABSTRACT. In this paper inequalities between two generalizations of the hyperbolic metric and the j_G metric are derived. We also prove inequalities between generalized versions of the j_G metric and Seittenranta's metric.

1. Introduction. In contrast to the situation in the complex plane, the well-known Poincaré hyperbolic metric is defined only in balls and half-spaces in $\mathbf{R}^n$ when $n \geq 3$. Many researchers have proposed metrics that could take the place of the hyperbolic metric in analysis in higher dimensions. Probably the most used one is the quasihyperbolic metric introduced by Gehring and Palka in [**5**]. This metric has the slight disadvantage in that it does not equal the hyperbolic metric in a ball, but rather may be off by a multiplicative constant of 2. For accurate estimates, for instance asymptotically sharp inequalities, this might pose a problem.

Several metrics have also been proposed that are generalizations of the hyperbolic metric in the sense that they equal the hyperbolic metric if the domain of definition is a ball or a half-space. Some examples are the Apollonian metric introduced by Beardon in [**2**], the Ferrand metric [**3**], the Kulkani-Pinkall metric [**8**] and Seittenranta's metric [**9**].

The generalizations of the hyperbolic metric studied in this paper are based on directly using two simple closed form formulae for the hyperbolic metric in balls. This approach was suggested by Vuorinen in [**11**] and it yields generalized hyperbolic metrics that have the desirable property that they equal the hyperbolic metric in balls and half-spaces in all dimensions.

In what follows all topological operations are with respect to $\overline{\mathbf{R}^n}$ (see Section 2; for further reference, e.g., [**11**]).

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