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GENERIC SUBIDEALS OF GRAPH IDEALS AND FREE RESOLUTIONS

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ABSTRACT. For a graph of an n-cycle Δ with Alexander dual Δ^*, we study the free resolution of a subideal $G(n)$ of the Stanley-Reisner ideal I_{Δ^*}. We prove that if $G(n)$ is generated by 3 generic elements of I_{Δ^*}, then the second syzygy module of $G(n)$ is isomorphic to the second syzygy module of $(x_1, x_2, \dots, x_n)$. A result of Bruns shows that there is always a 3-generated ideal with this property. We show that it can be chosen to have a particularly nice form.

1. Introduction and background. Let Δ be a cycle and Δ^* its Alexander dual. The Stanley-Reisner ideals of such graphs and their free resolutions have been studied by many people, such as in [**1, 2, 8, 9, 15, 16**]. In this paper we study the free resolution of a subideal $G(n)$ of I_{Δ^*} consisting of three generic elements of I_{Δ^*}. The study of these ideals led to the following observation, which is our main theorem.

Theorem 1. *Let $G(n)$ be as above and let $\mathrm{Syz}_2(G(n))$ be the module of second syzygies. Then the resolution of $\mathrm{Syz}_2(G(n))$ is the same as that of $\mathrm{Syz}_2((x_1, x_2, \dots, x_n))$.*

That is to say, the tails of the resolutions, i.e., the modules and maps in the later part of the complexes, of the ideals $G(n)$ and $(x_1, x_2, \dots, x_n)$ are identical. For example, in five variables the three generators of $G(5)$ are $\alpha = r_1cde + r_2ade + r_3abe + r_4bcd + r_5abc$, $\beta = s_1cde + s_2ade + s_3abe + s_4bcd + s_5abc$, and $\gamma = t_1cde + t_2ade + t_3abe + t_4bcd + t_5abc$. The minimal free resolution of $G(5)$ looks like

$$0 \longrightarrow R \xrightarrow{d_5} R^5 \xrightarrow{d_4} R^{10} \xrightarrow{\varphi_3} R^8 \xrightarrow{\varphi_2} R^3 \xrightarrow{\varphi_1} R$$

where the maps d_4 and d_5 are exactly the same as the ones for the resolution of (a, b, c, d, e).

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