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A NON-AUTOMATIC (!) APPLICATION OF GOSPER'S ALGORITHM EVALUATES A DETERMINANT FROM TILING ENUMERATION

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ABSTRACT. We evaluate the determinant $\det_{1\le i,j\le n} \times \left(\binom{x+y+j}{x-i+2j} - \binom{x+y+j}{x+i+2j}\right)$, which gives the number of lozenge tilings of a hexagon with cut off corners. A particularly interesting feature of this evaluation is that it requires the proof of a certain hypergeometric identity which we accomplish by using Gosper's algorithm in a nonautomatic fashion.

The purpose of this paper is to provide a direct evaluation of the determinant

$$\det_{1\le i,j\le n}\left(\binom{x+y+j}{x-i+2j} - \binom{x+y+j}{x+i+2j}\right). \tag{1}$$

This determinant arises in our study [**4**] on the enumeration of lozenge tilings of hexagons with cut off corners. For example, consider a hexagon with side lengths $x+n$, n, y, $x+n$, n, y, in cyclic order, and angles of 120° of which two adjacent corners are cut off as in Figure 1(a).[1] Figure 1(b) shows a lozenge tiling of this region, by which we mean a tiling by unit rhombi with angles of 60° and 120°, referred to as lozenges. The number of these lozenge tilings is given by the determinant (1). This is seen by converting the lozenge tilings into families $(P_1, P_2, \dots, P_n)$ of nonintersecting lattice paths consisting of

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[1] To be precise, from the top-left corner we cut off a (reversed) staircase of the form $(y-1, y-2, \dots, 1)$, meaning that the cut-off staircase consists of $y-1$ rhombi in the first row, $y-2$ rhombi in the second row, etc., and from the top-right corner we cut off a staircase of the form $(n-1, n-2, \dots, 1)$.