ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 2, Summer 2002

PICK FUNCTIONS RELATED TO THE GAMMA FUNCTION

CHRISTIAN BERG AND HENRIK L. PEDERSEN

ABSTRACT. We show that the function

$$f(z) = \frac{\log \Gamma(z+1)}{z \operatorname{Log} z},$$

holomorphic in the complex plane cut along the negative real axis, is a Pick function and we find its integral representation. We also show that various other related functions are Pick functions.

1. Introduction. The function

$$f(x) = \frac{\log \Gamma(x+1)}{x \log x}, \quad x > 0,$$

has attracted the attention of several authors, see [**3, 2**] and [**7**]. In [**4**], we proved that the reciprocal function $1/f$ has a holomorphic extension to the cut plane

$$\mathcal{A} = \mathbf{C} \setminus \left]-\infty, 0\right],$$

and that this extension is a Stieltjes transform. We also found its Stieltjes representation. As a corollary, we obtained that the restriction of f' to the positive real axis is completely monotone, thereby answering a question raised by Dimitar Dimitrov at the Fifth International Symposium on Orthogonal Polynomials, Special Functions and Applications held in Patras in September 1999. This result was thus obtained by considering the reciprocal function, and in the course of the proof we had to establish that the only zeros of the function $\log \Gamma$, defined below, in $\mathcal{A}$ are those at $z = 1$ and $z = 2$. The reciprocal of a Stieltjes function is a Pick function, so the result of [**4**] implies that f is a Pick function.

2000 AMS *Mathematics subject classification.* Primary 33B15, Secondary 30E20, 30E15.
Keywords and phrases. Gamma function, completely monotone function.
Received by the editors on August 17, 2000.