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THE DE LA VALLÉE POUSSIN THEOREM FOR VECTOR VALUED MEASURE SPACES

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ABSTRACT. The purpose of this paper is to extend the de la Vallée Poussin theorem to $\mathrm{cabv}(\mu, X)$, the space of measures defined in Σ with values in the Banach space X which are countably additive, of bounded variation and μ-continuous, endowed with the variation norm.

1. Introduction. The celebrated theorem of de la Vallée Poussin, VPT in brief, characterizes the Lebesgue uniform integrability in the following way.

Let $\mathcal{F}$ be a family of scalar measurable functions on a finite measure space (Ω, Σ, μ). Then the following are equivalents.

(i) $\sup_{f\in\mathcal{F}} \int_\Omega |f|\, d\mu < \infty$ and $\mathcal{F}$ is uniformly integrable, i.e., the set $\{\int_E |f|\, d\mu, f \in \mathcal{F}\}$ converges uniformly to zero in A if $\mu(E) \to 0$.

(ii) If $E_{nf} = \{\omega \in \Omega : |f(\omega)| > n\}$, then $\lim_{n\to\infty} \int_{E_{nf}} |f|\, d\mu = 0$, uniformly in $\mathcal{F}$.

(iii) There is a Young function Φ with the property that $\Phi(x)/x$ is an increasing function: $\lim_{x\to\infty}(\Phi(x)/x) = \infty$, and there is a constant $0 < C < \infty$ such that $\sup_{f\in\mathcal{F}} \int_\Omega \Phi(|f|)\, d\mu = C$.

The theorem of Dunford states that the uniform integrability of a subset K of $L_1(\mu)$ is equivalent to the relative weak compactness of K, and in [**1**, subsection 2.1] Alexopoulos gives more information on the structure of K in the corresponding Orlicz space $L_\Phi(\mu)$. The uniform integrability also is essential in the study of the relative weak compactness in $L_1(\mu, X)$, in fact every conditionally weakly compact subset of $L_1(\mu, X)$ is uniformly integrable, [**3**, Theorem IV]. The purpose of this paper is to extend the VPT to $\mathrm{cabv}(\mu, X)$. This result allows us to characterize, in terms of the Orlicz theory, a condition in $\mathrm{cabv}(\mu, X)$ which plays the role of the uniform integrability in $L_1(\mu, X)$.

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