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TWO-POINT DISTORTION THEOREMS FOR SPHERICALLY CONVEX FUNCTIONS

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ABSTRACT. One-parameter families of sharp two-point distortion theorems are established for spherically convex functions f, that is, meromorphic univalent functions f defined on the unit disk $\mathbf{D}$ such that $f(\mathbf{D})$ is a spherically convex subset of the Riemann sphere $\mathbf{P}$. These theorems provide for $a, b \in \mathbf{D}$ sharp lower bounds on $d_{\mathbf{P}}(f(a), f(b))$, the spherical distance between $f(a)$ and $f(b)$, in terms of $d_{\mathbf{D}}(a, b)$, the hyperbolic distance between a and b, and the quantities $(1-|a|^2)f^\sharp(a), (1-|b|^2)f^\sharp(b)$, where $f^\sharp = |f'|/(1+|f|^2)$ is the spherical derivative. The weakest lower bound obtained is an invariant form of a known growth theorem for spherically convex functions. Each of the two-point distortion theorems is necessary and sufficient for spherical convexity. These two-point distortion theorems are equivalent to sharp two-point comparison theorems between hyperbolic and spherical geometry on a spherically convex region Ω on $\mathbf{P}$. Each of these two-point comparison theorems characterize spherically convex regions.

1. Introduction. We begin by surveying the relatively brief history of two-point distortion theorems for univalent functions in order to set the stage for our work. The classical theory of univalent functions often deals with the family S of normalized $(g(0) = 0, g'(0) = 1)$ univalent functions g defined on the unit disk $\mathbf{D} = \{z : |z| < 1\}$. Sharp growth and distortion theorems for functions in S are well-known; these results are necessary but not sufficient for univalence. In 1978 Blatter [**1**] established a sharp two-point distortion theorem for non-normalized univalent functions f defined on $\mathbf{D}$ which is also sufficient for univalence. Blatter's result gives a sharp lower bound on the Euclidean distance $|f(a)-f(b)|$ in terms of $d_{\mathbf{D}}(a, b)$, the hyperbolic distance between a and b relative to $\mathbf{D}$, and the quantities $(1-|a|^2)|f'(a)|, (1-|b|^2)|f'(b)|$. Later, Kim and Minda [**4**] extended the method of Blatter and obtained a one-parameter family of sharp two-point distortion theorems that were both necessary and sufficient for univalence. An invariant version of the classical growth theorem for

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