ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 30, Number 2, Summer 2000

ON THE EQUATION $y^x \pm x^y = z^2$

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ABSTRACT. In this note, we find all solutions of the diophantine equation $y^x \pm x^y = z^2$ when $\gcd(x, y) = 1$, the product xy is even and $\min(x, y) > 1$. Namely, there is only the obvious solution $3^2 - 2^3 = 1$.

0. Introduction. The very special equation $y^x - x^y = 1$ was considered by Catalan [**2**], as a particular case of the famous Catalan's equation $y^n - x^m = 1$. Catalan claimed that the only solution is $(x, y) = (2, 3)$, but did not give any proof. Indeed, a proof was published by Moret-Blanc in 1876. As indicated in Ribenboim [**5**, p. 95], a short proof of this result can now be obtained using the theorems of V.A. Lebesgue (1850) and Ko Chao (1960). There are also special proofs by Rotkiewicz (1960) and Skandalis (1982), see Ribenboim's book for more details.

Here we consider the much more general equation $y^x - x^y = z^2$. To solve it is not easy. Our method is not completely classical; we use a combination of lower bounds of linear forms in p-adic and Archimedian logarithms. This combination enables us to get a good upper bound for y. This bound is small enough to completely solve the problem, thanks to a computer verification. We notice that without this idea the bounds reachable would be too large to permit this verification.

We assume that xy is even because we have no idea how to solve this equation when x and y are both odd. We assume also that x and y are coprime, but even if this is not so, our work is no less interesting.

Our result is the following.

Theorem. *The only positive solution of the diophantine equation*

$$y^x \pm x^y = z^2$$

Received by the editors on April 13, 1999.
Key words and phrases. Exponential diophantine equations, linear forms in logarithms.
1991 AMS *Mathematics Subject Classification.* 11D61, 11D72.