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A COMBINATORIAL IDENTITY OF SUBSET-SUM POWERS IN RINGS

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ABSTRACT. Escott showed that, for any set of n natural numbers, the sum of the kth powers of the sums of subsets of even cardinality is equal to the sum of the kth powers of the sums of subsets of odd cardinality for $k = 1, \dots, n-1$. We present a new proof of this fact which shows that this result holds in noncommutative rings as well.

The main application of Theorem 1 is to the Prouhet-Tarry-Escott problem, which is to determine, for each $d \in \mathbf{N}$, the least m such that there exist $(a_1, \dots, a_m) \in \mathbf{N}^m$ and $(b_1, \dots, b_m) \in \mathbf{N}^m$ not permutations of each other so that $\sum_{i=1}^m a_i^k = \sum_{i=1}^m b_i^k$ for all $k \le d$. (We use $\mathbf{N}$ to denote the set of natural numbers, and for every $n \in \mathbf{N}$ we use $\mathbf{n}$ to denote the set $\{1, \dots, n\}$.) In [**3**], this author describes in detail one method of applying Theorem 1 to the Prouhet-Tarry-Escott problem. For a thorough discussion of the Prouhet-Tarry-Escott problem, see Borwein and Ingall's recent paper [**1**].

Dorwart and Brown [**2**, p. 624] attribute Theorem 1 to Escott. Here we give a fuller presentation of the old proof sketched by Borwein and Ingalls following their Proposition 1 in [**1**]. This proof has similarities to the one presented by Wright [**4**]. This proof shows that the theorem holds for natural numbers, and we follow it with a proof that the identity holds in noncommutative rings as well.

Theorem 1. *For any $n \in \mathbf{N}$ and $\alpha_1, \dots, \alpha_n \in \mathbf{N}$,*

$$\sum_{\substack{I \subseteq \mathbf{n} \\ |I| \text{ odd}}} \left(\sum_{i \in I} \alpha_i \right)^k = \sum_{\substack{I \subseteq \mathbf{n} \\ |I| \text{ even}}} \left(\sum_{i \in I} \alpha_i \right)^k,$$

for all $k < n$.

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