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DIOPHANTINE TRIPLES AND CONSTRUCTION OF HIGH-RANK ELLIPTIC CURVES OVER Q WITH THREE NONTRIVIAL 2-TORSION POINTS

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1. Introduction. Let E be an elliptic curve over $\mathbf{Q}$. The famous theorem of Mordell-Weil states that

$$E(\mathbf{Q}) \simeq E(\mathbf{Q})_{\text{tors}} \times \mathbf{Z}^r,$$

and by a theorem of Mazur [**15**] we know that only possible torsion groups over $\mathbf{Q}$ are

$$E(\mathbf{Q})_{\text{tors}} = \begin{cases} \mathbf{Z}/m\mathbf{Z} & m = 1, 2, \dots, 10 \text{ or } 12, \\ \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2m\mathbf{Z} & m = 1, 2, 3, 4. \end{cases}$$

Let

$$B(F) = \sup\{\text{rank}\,(E) : E \text{ curve over } \mathbf{Q} \text{ with } E(\mathbf{Q})_{\text{tors}} \simeq F\},$$
$$B_r(F) = \limsup\{\text{rank}\,(E) : E \text{ curve over } \mathbf{Q} \text{ with } E(\mathbf{Q})_{\text{tors}} \simeq F\}.$$

An open question is whether $B(F) < \infty$.

The examples of Martin-McMillen and Fermigier [**8**] show that $B(0) \geq 23$ and $B(\mathbf{Z}/2\mathbf{Z}) \geq 14$. It follows from results of Montgomery [**18**] and Atkin-Morain [**1**] that $B_r(F) \geq 1$ for all torsion groups F. Kihara [**11**] proved that $B_r(0) \geq 14$ and Fermigier [**8**] that $B_r(\mathbf{Z}/2\mathbf{Z}) \geq 8$. Recently, Kihara [**12**] and Kulesz [**14**] proved using parametrization by $\mathbf{Q}(t)$ and $\mathbf{Q}(t_1, t_2, t_3, t_4)$ that $B_r(\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}) \geq 4$ and Kihara [**13**] proved using parametrization by rational points of an elliptic curve that $B_r(\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}) \geq 5$. Kulesz also proved that $B_r(\mathbf{Z}/3\mathbf{Z}) \geq 6$, $B_r(\mathbf{Z}/4\mathbf{Z}) \geq 3$, $B_r(\mathbf{Z}/5\mathbf{Z}) \geq 2$, $B_r(\mathbf{Z}/6\mathbf{Z}) \geq 2$ and $B_r(\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}) \geq 2$. The methods used in [**12**] and [**14**] are similar to the method of Mestre [**16, 17**].

In the present paper we prove that $B_r(\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}) \geq 4$ by a different method. Namely, we use the theory of, so called, Diophantine

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