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PERTURBATION OF FRAMES FOR A SUBSPACE OF A HILBERT SPACE

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ABSTRACT. A frame sequence $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ allows every element in the closed linear span, $[f_i]$, to be written as an infinite linear combination of the frame elements f_i. Thus a frame sequence can be considered to be some kind of "generalized basis." Using an extension of a classical condition, we prove that a perturbation $\{g_i\}_{i=1}^{\infty}$ of a frame sequence $\{f_i\}_{i=1}^{\infty}$ is again a frame sequence whenever the gap from $[g_i]$ to $[f_i]$ is small enough. In the special case of a Riesz sequence $\{f_i\}_{i=1}^{\infty}$ the gap condition may be omitted.

1. Introduction. A frame sequence $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ has the property that every element in $[f_i] := \overline{\text{span}}\,\{f_i\}_{i=1}^{\infty}$ has a representation as an infinite linear combination of the frame elements f_i. In contrast with the situation for a basis, the corresponding coefficients are not necessarily unique, which makes frame sequences a very useful tool when more freedom is required. A frame sequence is thus a very natural generalization of the concept of a Riesz sequence (i.e., a sequence that is a Riesz basis for its closed linear span).

Our goal is to prove some perturbation results for frame sequences. To motivate the following, remember that if $\{f_i\}_{i=1}^{\infty}$ is a Riesz sequence, then $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a Riesz sequence whenever

$$(1) \qquad \bigg\| \sum_{i=1}^{\infty} c_i (f_i - g_i) \bigg\| \le \mu \bigg(\sum_{i=1}^{\infty} |c_i|^2 \bigg)^{1/2}, \quad \forall \{c_i\}_{i=1}^{\infty} \in l^2(N),$$

for a sufficiently small constant μ. We prove the same conclusion holds under a weaker condition than (1).

The direct analogue of this last Riesz sequence result for a frame sequence $\{f_i\}_{i=1}^{\infty}$ does not hold unless $[f_i]$ is the whole Hilbert space. This leads us to consider the notion of the gap from one subspace of

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