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DIOPHANTINE APPROXIMATION BY CUBES OF PRIMES AND AN ALMOST PRIME

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ABSTRACT. Let $\lambda_1, \dots, \lambda_s$ be nonzero with λ_1/λ_2 irrational, and let $\mathcal{S}$ be the set of values attained by the form

$$\lambda_1 x_1^3 + \cdots + \lambda_s x_s^3$$

when x_1 has at most six prime divisors and the remaining variables are prime. In the case $s = 4$, we establish that most real numbers are "close" to an element of $\mathcal{S}$. We then prove that if $s = 8$, $\mathcal{S}$ is dense on the real line.

1. Introduction and preliminaries. Let $\lambda_1, \dots, \lambda_s$ be nonzero real numbers with λ_1/λ_2 irrational, and let $\mathbf{P}^s$ denote the set of integer points in $\mathbf{R}^s$, all coordinates of which are prime. We will be concerned with the distribution of the values taken by the form

$$\lambda_1 x_1^3 + \cdots + \lambda_s x_s^3 \tag{1.1}$$

on $\mathbf{P}^s$. The (optimistic) conjecture is that if $s \geq 4$, they are dense on $\mathbf{R}$, but our factual knowledge on the topic is much worse. Back in 1963, Schwarz [**13**] showed that if $s \geq 9$, the values of (1.1) on $\mathbf{P}^s$ are dense, and although sharper quantitative versions of this result have been obtained (see Vaughan [**16**] and Baker and Harman [**1**]), it seems that reducing the minimum value of s is beyond the limit of the present methods. On the other hand, in the similar situation with the classical Waring-Goldbach problem for cubes, Roth [**11**] showed that if one allows x to take arbitrary integer values, the equation

$$x^3 + p_1^3 + p_2^3 + p_3^3 = n \tag{1.2}$$

is solvable for almost all integers n, in the sense usually adopted in additive number theory. Let $\mathcal{P}_r$ denote the set of integers having at most r prime divisors counted with multiplicities. Brüdern [**3**] proved

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