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A NOTE ON SCHUR-CONVEX FUNCTIONS

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ABSTRACT. In this note it is proved that the integral arithmetic mean of a convex function is a Schur-convex function. Applications to Schur-convexity of logarithmic mean and gamma functions are given.

For the convenience of the reader, we recall shortly the main definitions. Function F of n arguments defined on I^n, where I is an interval with nonempty interior, is Schur-convex on I^n if

$$F(x_1, \dots, x_n) \le F(y_1, \dots, y_n) \tag{1}$$

for each two n-tuples $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ in I^n, such that $x \prec y$ holds, i.e.,

$$\begin{aligned} \sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}, \quad k = 1, \dots, n-1, \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}, \end{aligned} \tag{2}$$

where $x_{[i]}$ denotes the ith largest component in x. F is strictly Schur-convex on I^n if a strict inequality holds in (1) whenever $x \prec y$ and x is not a permutation of y.

For $n = 2$, a continuously differentiable function F on I^2 (I being an open interval) is Schur-convex if and only if it is symmetric and the following holds

$$\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right)(y - x) > 0 \quad \text{for all } x, y \in I,\ x \ne y. \tag{3}$$

Of course, F is Schur-concave if and only if $-F$ is Schur-convex.

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