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WILLMORE TORI IN A WIDE FAMILY OF CONFORMAL STRUCTURES ON ODD DIMENSIONAL SPHERES

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ABSTRACT. We obtain a variable reduction principle for the Willmore variational problem in an ample class of conformal structures on S^{2n+1}. This variational problem is transformed into another one, associated with an elastic-energy functional with potential, on spaces of curves in CP^n. Then, we give a simple method to construct Willmore tori in certain conformal structures on S^{2n+1}. Moreover, we exhibit some families of Willmore tori for the standard conformal class on S^3 and S^7.

1. Introduction. Let $\mathbf{S}^{2n+1}$ be the unit sphere in $\mathbf{C}^{n+1}$ endowed with the standard metric $\bar{g}$. The unit circle $\mathbf{S}^1$ acts naturally on $\mathbf{S}^{2n+1}$ to produce $\mathbf{C}P^n$ as orbit space. The canonical projection $\pi : (\mathbf{S}^{2n+1}, \bar{g}) \rightarrow (\mathbf{C}P^n, g)$ is a Riemannian submersion, where g denotes the Fubini-study metric of constant holomorphic sectional curvature 4. A vertical, unit global vector field V is defined on $\mathbf{S}^{2n+1}$ by $V(z) = iz$, for all $z \in \mathbf{S}^{2n+1}$. The horizontal distribution $\mathcal{H}$ is defined to be the $\bar{g}$-orthogonal complementary to the orbits. As usual, overbars will denote horizontal lifts of the corresponding objects in a Riemannian submersion (see [**6**], [**13**] for details about notation and terminology). In particular, the Levi-Civita connections $\bar{\nabla}$ and ∇ of $\bar{g}$ and g, respectively, are related via the following well-known formulae:

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \overline{\nabla_X Y} - \bar{g}(i\bar{X}, \bar{Y})V, \tag{1.1}$$

$$\bar{\nabla}_{\bar{X}}V = \bar{\nabla}_V \bar{X} = i\bar{X}, \tag{1.2}$$

$$\bar{\nabla}_V V = 0. \tag{1.3}$$

Remark 1. (i) It should be noticed that the last formula shows the geodesic nature of the orbits in $(\mathbf{S}^{2n+1}, \bar{g})$. (ii) Since π may also be

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