ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 2, Summer 2001

A GENERALIZATION OF A THEOREM OF COHN ON THE EQUATION $x^3 - Ny^2 = \pm 1$

F. LUCA AND P.G. WALSH

1. Introduction. In [**2**], Cohn investigated the solvability of the Diophantine equation

$$x^3 - Ny^2 = \pm 1. \tag{1.1}$$

Improving upon previous work of Stroeker [**5**], Cohn proved the following theorem.

Theorem A. *Let N denote a squarefree positive integer with no prime factor of the form $3k+1$. Then the equation $x^3 - Ny^2 = 1$ has no solutions in positive integers, and the equation $x^3 - Ny^2 = -1$ has no solutions in positive integers, unless $N \in \{1, 2\}$, in which case $(N, x, y) = (1, 2, 3)$ and $(N, x, y) = (2, 23, 78)$ are the only solutions.*

The interesting case in this theorem arises when the irreducible quadratic factors of $x^3 \pm 1$ take on values of the form $3z^2$, for otherwise the result is an immediate consequence of quadratic reciprocity. Cohn deals with this case in a very clever manner by determining all of the integer solutions to the respective equations

$$x^2 + x + 1 = 3z^2, \qquad x - 1 = 3Nw^2$$

and

$$x^2 - x + 1 = 3z^2, \qquad x + 1 = 3Nw^2,$$

which are equivalent respectively to

$$3N^2w^4 + 3Nw^2 + 1 = z^2 \tag{1.2}$$

1991 AMS *Mathematics Subject Classification.* 11D25, 11J86.
Received by the editors on October 22, 1999, and in revised form on December 7, 1999.