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A COEFFICIENT PROBLEM FOR UNIVALENT FUNCTIONS RELATED TO TWO-POINT DISTORTION THEOREMS

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ABSTRACT. We discuss a class of coefficient functionals over the set of normalized univalent functions on the unit disk. These functionals are related to symmetric, linearly invariant two-point distortion theorems for univalent functions due to Kim and Minda. Each of these theorems is necessary and sufficient for univalence. A special case is a distortion theorem of Blatter. Our approach is based on an application of Pontryagin's maximum principle to the Loewner differential equation. In the same fashion, two-point distortion theorems for bounded univalent functions are obtained. Related coefficient functionals are discussed, too.

1. Introduction. Let $\mathcal{S}$ be the customary class of normalized univalent functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

on the unit disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ into $\mathbf{C}$, and consider for a fixed number $p \in \mathbf{R}$ the functional $J_p : \mathcal{S} \to \mathbf{R}$ defined by

$$(1) \qquad J_p(f) = J_p(a_2, a_3) := \operatorname{Re}\left(a_3 + \frac{p-3}{3}{a_2}^2\right) + \frac{p+1}{3}|a_2|^2.$$

Every function $F \in \mathcal{S}$ maximizing J_p over $\mathcal{S}$ is called an *extremal function* for J_p.

The coefficient functional J_p is related to the following one-parameter family of symmetric, linearly invariant two-point distortion theorems for (not necessarily normalized) univalent functions on the unit disk due to Blatter [**1**] and Kim and Minda [**7**].

Theorem 1.1. *Let $p > 0$ be a real number such that the Koebe function $K(z) := z/(1-z)^2 \in \mathcal{S}$ maximizes the functional* (1) *over the*

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