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FACTORIZATION IN COMMUTATIVE RINGS WITH ZERO DIVISORS, III

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ABSTRACT. Let R be a commutative ring with identity. We continue our study of factorization in commutative rings with zero divisors. In Section 2 we consider inert extensions and atomicity. In Section 3 we characterize the atomic rings in which almost all atoms are prime. In Section 4 we investigate bounded factorization rings (BFR's) and U-BFR's, and in Section 5 we study finite factorization rings (FFR's).

1. Introduction. Throughout this paper, R will be a commutative ring with identity. This article is the third in a series of papers [**10**], [**11**] considering factorization in commutative rings with zero divisors. Here we concentrate on atomic rings, especially bounded factorization rings and finite factorization rings, which are defined below. We first review the various forms of irreducible elements introduced in [**10**].

For an integral domain R, a nonzero nonunit $a \in R$ is said to be irreducible or to be an atom if $a = bc$, $b, c \in R$, implies b or $c \in U(R)$, the group of units of R. It is easily checked that a is an atom $\Leftrightarrow$ (a) is a maximal (proper) principal ideal of $R \Leftrightarrow a = bc$ implies b or c is an associate of a. Now if R has zero divisors, these various characterizations of being irreducible no longer need to be equivalent. The following different forms of irreducibility are based on elements being associates. Let $a, b \in R$. Then a and b are *associates*, denoted $a \sim b$ if $a|b$ and $b|a$, i.e., $(a) = (b)$, a and b are *strong associates*, denoted $a \approx b$, if $a = ub$ for some $u \in U(R)$, and a and b are *very strong associates*, denoted $a \cong b$, if $a \sim b$ and either $a = 0$ or $a = cb$ implies $c \in U(R)$. Then a nonunit $a \in R$ (possibly with $a = 0$) is *irreducible* (respectively, *strongly irreducible, very strongly irreducible*), if $a = bc \Rightarrow a \sim b$ or $a \sim c$, respectively $a \approx b$ or $a \approx c$, $a \cong b$ or $a \cong c$. And a is *m-irreducible* if (a) is maximal in the set of proper principal ideals of R. A nonzero nonunit $a \in R$ is very strongly irreducible $\Leftrightarrow a = bc$ implies b or $c \in U(R)$ [**10**, Theorem 2.5]. Now a is very

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