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# ON THE FREDHOLM INDICES OF ASSOCIATED SYSTEMS OF WIENER-HOPF EQUATIONS

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ABSTRACT. Every matrix function $A \in L^\infty_{n\times n}(\mathbf{R})$ generates a Wiener-Hopf integral operator on $L^2_n(\mathbf{R}_+)$, the direct sum of $n$ copies of $L^2(\mathbf{R}_+)$. The associated Wiener-Hopf integral operator is the operator $W(\tilde{A})$ where $\tilde{A}(x) := A(-x)$. We discuss the connection between the Fredholm indices $\operatorname{Ind} W(A)$ and $\operatorname{Ind} W(\tilde{A})$. Our main result says that if $A$ has at most a finite number $d$ of discontinuities on $\mathbf{R}\cup\{\infty\}$ and both $W(A)$ and $W(\tilde{A})$ are Fredholm, then

$$|\operatorname{Ind} W(A) + \operatorname{Ind} W(\tilde{A})| \le d(n-1);$$

conversely, given integers $\kappa$ and $\nu$ satisfying $|\kappa+\nu| \le d(n-1)$, there exist $A \in L^\infty_{n\times n}(\mathbf{R})$ with at most $d$ discontinuities such that $W(A)$ is Fredholm of index $\kappa$ and $W(\tilde{A})$ is Fredholm of index $\nu$.

**1. Introduction and main results.** Given a measurable subset $\Omega$ of the real line $\mathbf{R}$, we denote by $L^p_{n\times n}(\Omega)$ and $L^p_n(\Omega)$ the $n\times n$ matrix functions with entries in $L^p(\Omega)$ and the column vectors of height $n$ with components in $L^p(\Omega)$, respectively. For $A \in L^\infty_{n\times n}(\mathbf{R})$, the convolution operator with the symbol $A$ is the operator

$$C(A) : L^2_n(\mathbf{R}) \longrightarrow L^2_n(\mathbf{R}), \qquad f \longmapsto \mathcal{F}^{-1}A\mathcal{F}f,$$

where $\mathcal{F}$ is the Fourier transform,

$$(\mathcal{F}f)(x) := \int_{\mathbf{R}} f(t)e^{ixt}\,dt, \quad x \in \mathbf{R}.$$

Let $\mathbf{R}_+ = (0,\infty)$. The compression of $C(A)$ to $L^2_n(\mathbf{R}_+)$ is referred to as the Wiener-Hopf operator with the symbol $A$ and will be denoted by $W(A)$. Thus,

$$W(A) : L^2_n(\mathbf{R}_+) \longrightarrow L^2_n(\mathbf{R}_+), \quad f \longmapsto P\mathcal{F}^{-1}A\mathcal{F}f,$$

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