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PLANAR REAL POLYNOMIAL DIFFERENTIAL SYSTEMS OF DEGREE $n > 3$ HAVING A WEAK FOCUS OF HIGH ORDER

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ABSTRACT. We construct planar polynomial differential systems of even (respectively odd) degree $n > 3$, of linear form plus a nonlinear homogeneous part of degree n having a weak focus of order $n^2 - 1$, respectively $(n^2 - 1)/2$, at the origin. As far as we know this provides the highest order known until now for a weak focus of a polynomial differential system of arbitrary degree n.

1. Introduction and statement of the main result. For every $\alpha \in \mathbf{R}$ we consider a real homogeneous polynomial $f_\alpha(x, y)$ of degree $n - 1$ and the following real polynomial differential system

$$\frac{dx}{dt} = \dot{x} = -y(1 - f_\alpha(x, y)), \qquad \frac{dy}{dt} = \dot{y} = x(1 - f_\alpha(x, y)),$$

which has the algebraic curve $\{f_\alpha = 1\}$ of singular points, and an isolated singularity at the origin, i.e., $f_\alpha(0, 0) \neq 1$. We perturb this system as follows

$$\begin{aligned} \dot{x} &= -y(1 - f_\alpha(x, y)) + P(x, y), \\ \dot{y} &= x(1 - f_\alpha(x, y)) + Q(x, y), \end{aligned} \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are homogeneous polynomials of degree $n > 3$ with small real coefficients.

It is well known that system (1) always has either a center or a weak focus at the origin, i.e., a monodromic singularity, see for instance [**1**,

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