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REMARKS ON GENERALIZED TRIGONOMETRIC FUNCTIONS

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ABSTRACT. A natural generalization of the sine function occurs as an eigenfunction of the Dirichlet problem for the one-dimensional p-Laplacian. Our study of the properties of p-trigonometric functions and their connection with classical analysis leads to a variety of new identities and inequalities and to the basis properties of the p-eigenfunctions.

1. Introduction. The spectral properties of the Dirichlet Laplacian on the unit interval of the real line are familiar and simple: the problem

$$-u'' = \lambda u \text{ on } (0,1), \quad u(0) = u(1) = 0 \tag{1.1}$$

has eigenvalues $(n\pi)^2$ and corresponding eigenvectors u_n, $u_n(t) = \sin(n\pi t)$ ($n \in \mathbf{N}$). It is a remarkable fact (see, for example, [**6**]) that the corresponding problem for the one-dimensional p-Laplacian Δ_p ($1 < p < \infty$), namely,

$$\begin{aligned} -\Delta_p u := -\left(|u'|^{p-2}\, u'\right)' = \lambda\, |u|^{p-2}\, u \quad \text{on } (0,1), \\ u(0) = u(1) = 0, \end{aligned} \tag{1.2}$$

has eigenfunctions expressible in terms of functions similar to the sine function. In fact, (1.2) has eigenvalues

$$\lambda_n = (p-1)(n\pi_p)^p, \quad \text{where } \pi_p = \frac{2\pi}{p\sin(\pi/p)},$$

and associated eigenfunctions $\sin_p(n\pi_p t)$ ($n \in \mathbf{N}$). Here $\sin_p$ is the function defined on $[0, \pi_p/2]$ to be the inverse of the function $F_p : [0,1] \to \mathbf{R}$ given by

$$F_p(x) = \int_0^x (1-t^p)^{-1/p} dt,$$

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