

# ON REPRESENTATIONS AND DIFFERENCES OF STIELTJES COEFFICIENTS, AND OTHER RELATIONS

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**ABSTRACT.** The Stieltjes coefficients  $\gamma_k(a)$  arise in the expansion of the Hurwitz zeta function  $\zeta(s, a)$  about its single simple pole at  $s = 1$  and are of fundamental and long-standing importance in analytic number theory and other disciplines. We present an array of exact results for the Stieltjes coefficients, including series representations and summatory relations. Other integral representations provide the difference of Stieltjes coefficients at rational arguments. The presentation serves to link a variety of topics in analysis and special function and special number theory, including logarithmic series, integrals, and the derivatives of the Hurwitz zeta and Dirichlet  $L$ -functions at special points. The results have a wide range of application, both theoretical and computational.

**1. Introduction and statement of results.** The Stieltjes (or generalized Euler) constants  $\gamma_k(a)$  appear as expansion coefficients in the Laurent series about  $s = 1$  for the Hurwitz zeta function  $\zeta(s, a)$ , one of the generalizations of the Riemann zeta function  $\zeta(s)$ . Elsewhere [6], we developed new summatory relations amongst the values  $\gamma_k(a)$  as well as demonstrated one of the very recent conjectures put forward by Kreminski [19] on the relationship between  $\gamma_k(a)$  and  $-\gamma_k(a + 1/2)$  as  $k \rightarrow \infty$  [5]. New series representations of the Riemann and Hurwitz zeta functions, as well as series representations of  $\gamma_1(a)$  and  $\gamma_2(a)$  are given very recently in [9].

In this paper, we present an array of exact results for the Stieltjes constants. These include individual and summatory relations for paired differences of these coefficients for rational arguments. Our work provides a unification of several important topics of analysis and analytic number theory. These include certain logarithmic sums, integrals of

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