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GREEN'S FUNCTION FOR THIRD-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. Existence of positive periodic solutions of third-order differential equations is established using an explicit Green's function and a fixed-point theorems on cones.

1. Introduction. The study of periodic solutions plays a major role in the theory of differential equations, in particular the existence of positive periodic solutions of these equations. Currently there are plenty of existence results for periodic solutions (or positive periodic solutions) for second-order differential equations, but there are relatively few results on higher order differential equation. On the other hand, in the study of higher order differential equations, the naive idea to translate the equation into a first order system of differential equations by defining $x_1 = x$, $x_2 = x'$, $x_3 = x'', \dots$, which works well for showing existence of periodic solutions, does not obviously lead to existence proofs for positive periodic solutions, since the condition $x = x_1 \geq 0$ of positivity for the higher order equation is different from the natural positivity condition $(x_1, x_2, \dots) \geq 0$ for the corresponding system. An approach which is frequently used is to transform the higher-order equation into a corresponding integral equation and to establish the existence of positive periodic solutions based on a fixed point theorem in cones. Following this path one needs an explicit representation of the Green's function which is rather intricate to compute. In this paper, we provide the Green's function for third-order differential equations with constant coefficients. This should be helpful for further studies of this type of equations.

The remaining part of the paper is organized as follows. In Section 2, the Green's function for third-order constant-coefficient linear differential equation

$$u''' + au'' + bu' + cu = h(t), \tag{1.1}$$

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