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A SURVEY ON WEIGHTED DENSITIES AND THEIR CONNECTION WITH THE FIRST DIGIT PHENOMENON

BRUNO MASSÉ AND DOMINIQUE SCHNEIDER

ABSTRACT. This paper is a general treatment of the various notions of densities used in papers on mantissa distribution of sequences of numbers. Equivalence classes of weighted densities are identified, and their hierarchy is stated. This permits us to give clear answers to several questions about the first digit phenomenon. Moreover, however light the weights are, we exhibit an example of a sequence of positive numbers whose mantissae do not admit any distribution in the sense of the corresponding density.

1. Introduction and definitions. Following the early works of Benford and Newcomb [**1, 17**] on real life numbers, many authors have studied the distribution of the first digit in base 10 of sequences $(u_n)_n$ of positive numbers like $u_n = 2^n$, $u_n = n!$, $u_n = n^n$ and $u_n = F_n$ where F_n is the nth Fibonacci number, $u_n = n$ or $u_n = p_n$ where p_n is the nth prime number and so on (see [**19**] for a survey). In the first four cases they proved that, if $D(u_n)$ denotes the first digit of u_n and $\log_{10}$ the decimal logarithm, the natural density of $A_k^u = \{n \in \mathbf{N}^* : D(u_n) = k\}$ is $\log_{10}((k+1)/k)$, that is to say,

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{A_k^u}(n) = \log_{10}\left(\frac{k+1}{k}\right) \quad (k = 1, \dots, 9);$$

(here and in the sequel, $\mathbf{1}_B$ is the indicator function of the subset B). In particular, about 30.1 percent of the u_n have first digit 1 in the sense of the above formula. This property is known as the *first digit phenomenon.* Classical applications of this phenomenon are fraud detection [**18**] and computer design [**11, 15**].

In fact, we know a more precise property which needs three other definitions to be stated: *Benford's law* (in base 10) is the probability

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