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ON KRONECKER POLYNOMIALS

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ABSTRACT. Monic polynomials with integer coefficients having all their roots in the unit disc have been studied by Kronecker; they are called *Kronecker polynomials*. Let $n \geq 1$ be an integer. By a *strong Kronecker polynomial*, we mean a monic polynomial $P(X) \in \mathbf{Z}[X]$ of degree $n-1$ and such that $P(X)$ divides $P(X^t)$ for each $t \in \{1, \dots, n-1\}$. We say that $P(X)$ is an *absolutely Kronecker polynomial* if $P(X)$ divides $P(X^t)$ for each positive integer t. We describe a canonical form of strong (respectively absolute) Kronecker polynomials. We, also, prove that if n is composite, then each strong Kronecker polynomial with degree $n-1$ is absolutely Kronecker. If n is prime, then we prove that each strong Kronecker polynomial $P(X) \neq 1 + X + X^2 + \dots + X^{n-1}$ is absolutely Kronecker.

0. Introduction. In 1857, Kronecker [**4**] was interested in monic polynomials (i.e., with highest coefficient 1) with integer coefficients having all their roots in the unit disc (Kronecker polynomials). Kronecker proved that the non-zero roots of such polynomials are on the boundary of the unit disc (the unit circle); he also proved that there are finitely many such polynomials of degree a given positive integer n.

In 2001, Pantelis Damianou [**3**] described a canonical form of these polynomials and called them *Kronecker polynomials*. He proved that these polynomials have the form $P(X) = X^k Q(X)$, where $Q(X)$ is a finite product of cyclotomic polynomials.

In 2000, Doru Caragea and Viviana Ene proposed the following "Millennial polynomial problem" [**1**]: Let S be the set of monic, irreducible polynomials with degree 2000 and integer coefficients. Find all $P \in S$ such that $P(a)$ divides $P(a^2)$ for every natural number a.

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