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ON FINITE SUMS OF LAGUERRE POLYNOMIALS

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ABSTRACT. We evaluate a family of finite summations of Laguerre polynomials in terms of Laguerre polynomials themselves or in terms of a generalized hypergeometric function. As a byproduct, we find a Kummer-like transformation of the hypergeometric function ${}_2F_2$ in terms of a Laguerre polynomial. The results have applications in the theory of special functions as well as in analytic number theory.

0. Introduction. In recent years it has become evident that the Laguerre calculus [**3**] is important in the description of the Li criterion [**11**] for the Riemann hypothesis (RH) [**5, 6**]. The Laguerre polynomials and their properties also play an important role in many areas of mathematical physics, including random matrix theory, Fourier optics, and quantum mechanics. The Laguerre polynomials L_n^α appear prominently in the solution of the higher dimensional Kepler-Coulomb and harmonic oscillator problems, whose wavefunctions constitute "quantum shapelets" [**7**].

Since the Laguerre polynomials form a Sheffer sequence with a special generating function of exponential form, these polynomials are singled out in developing representations of special functions under fractional linear transformation [**6**]. In particular, the Laguerre polynomials have properties that are crucial in formulating recurrence and integral relations for transformations of the Riemann xi function, and therefore for describing the Li criterion. It appears that the particular Laguerre polynomials L_{n-1}^1 are very important as test functions for a Weil inner product whose nonnegativity is equivalent to the RH.

Given these several reasons to further investigate the classical orthogonal Laguerre polynomials [**4, 13**], we consider here finite sums that

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