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THE BOOLEAN SPACE OF R-PLACES

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ABSTRACT. We prove that every Boolean space is realized as a space of real places of some formally real field. This gives a partial answer to the problem posed in [**1, 7**].

1. Introduction. Let $\mathcal{X}(K)$ be the space of orders of a formally real field K, endowed with the Harrison topology introduced by subbasic sets of the form

$$H_K(a) := \{P \in \mathcal{X}(K) : a \in P\}, \quad a \in \dot{K} = K \setminus \{0\}.$$

It is known that $\mathcal{X}(K)$ is a Boolean space, i.e., compact, Hausdorff and totally disconnected. In [**3**] Craven presented a construction of a field K, whose space of orders $\mathcal{X}(K)$ is homeomorphic to a given Boolean space X. Spaces of orders are closely related to the spaces of $\mathbf{R}$-places, and some main results on this relationship can be found in [**8**]. We shall recall a part of this theory in the next section. In particular, spaces of $\mathbf{R}$-places are known to be compact and Hausdorff. An open problem posed in [**1, 7**] is:

Which compact and Hausdorff spaces occur as a spaces of real places?

It was pointed out in [**1**, Remark 2.16] that if K is a totally Archimedean field then the space of $\mathbf{R}$-places and the space of orders are homeomorphic and consequently the space of $\mathbf{R}$-places is Boolean. Thus, every finite discrete space is realized as a space of $\mathbf{R}$-places, since totally Archimedean fields exist with any finite number of orders. Our main theorem, presented in Section 4, states that every Boolean space is realized as a space of $\mathbf{R}$–places of some formally real field. Before we can get to this, we need to develop some new methods in the theory of extensions of $\mathbf{R}$-places; Section 3 includes these results.

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