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ARITHMETIC PROGRESSIONS IN THE SOLUTION SETS OF NORM FORM EQUATIONS

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1. Introduction. Let K be an algebraic number field of degree k, and let $\alpha_1, \dots, \alpha_n$ be linearly independent elements of K over $\mathbf{Q}$. Denote by $D \in \mathbf{Z}$ the common denominator of $\alpha_1, \dots, \alpha_n$, and put $\beta_i = D\alpha_i$, $i = 1, \dots, n$. Note that $\beta_1, \dots, \beta_n$ are algebraic integers of K. Let m be a nonzero integer, and consider the norm form equation

$$N_{K/\mathbf{Q}}(x_1\alpha_1 + \cdots + x_n\alpha_n) = m \tag{1.1}$$

in integers $x_1, \dots, x_n$. Let H denote the solution set of (1.1) and $|H|$ the size of H. Note that, if the $\mathbf{Z}$-module generated by $\alpha_1, \dots, \alpha_n$ contains a submodule, which is a full module in a subfield of $\mathbf{Q}(\alpha_1, \dots, \alpha_n)$ different from the imaginary quadratic fields and $\mathbf{Q}$, then this equation can have infinitely many solutions (see, e.g., Schmidt [**19**]). Various arithmetical properties of the elements of H were studied in [**8, 11**]. In the present paper we are concerned with arithmetical progressions in H. Arranging the elements of H in an $|H| \times n$ array $\mathcal{H}$, one may ask at least two natural questions about arithmetical progressions appearing in H. The "horizontal" one: do there infinitely many rows of $\mathcal{H}$ exist, which form arithmetic progressions; and the "vertical" one: do arbitrary long arithmetic progressions in some column of $\mathcal{H}$ exist? Note that the first question is meaningful only if $n > 2$.

The "horizontal" problem was treated by Bérczes and Pethő [**4**] by proving that if $\alpha_i = \alpha^{i-1}$, $i = 1, \dots, n$, then in general $\mathcal{H}$ contains only finitely many effectively computable "horizontal" APs, and they were able to localize the possible exceptional cases. Later Bérczes and Pethő [**5**], Bérczes Pethő and Ziegler [**6**] and Bazsó [**2**]

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