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THE WEYL CORRESPONDENCE AS A FUNCTIONAL CALCULUS FOR NON-COMMUTING OPERATORS

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ABSTRACT. In this expository paper, we describe the Weyl calculus for bounded, self-adjoint operators acting on a Hilbert space as well as the original Weyl correspondence for the position Q and momentum P operators on $S(\mathbf{R}^n)$. We describe some classes of functions for which the calculus is well defined and give a representation for the action of the calculus in these separate cases. In particular, we verify that the Weyl calculus is well defined for polynomials and give results consistent with the natural algebraic definition. The proof of this result for the original Weyl correspondence is obtained via an analysis of the commutator of P and Q on $S(\mathbf{R}^{2n})$. We also discuss the connection of the Weyl calculus with some recent developments in functional calculi.

1. Introduction. The Weyl correspondence was developed in efforts by Hermann Weyl and other mathematical physicists to better understand the correlation between physical observables in classical mechanics and their quantum-mechanical analogues. In classical mechanics, one is concerned with the state space $(p, q) \in \mathbf{R}^{2n}$ that represents the momentum and position information of an object. The observables come in the form of real-valued functions f defined on this space, such as the position operator $(p, q) \mapsto q_j$ and the momentum $(p, q) \mapsto p_j$. In the quantum-mechanical view, this state space is replaced by the set of functions $f \in L^2(\mathbf{R}^n)$ for which $\|f\|_2 = 1$, i.e., the *wavefunctions*, and the observables become self-adjoint linear operators A acting on this space. The fundamental questions to be answered are: how does one connect the state space $\mathbf{R}^{2n}$ to the set of wavefunctions in $L^2(\mathbf{R}^n)$ and, given this, what is the correlation between the real-valued functions $f(p, q)$ and the self-adjoint linear operators A?

The motivation for the Weyl calculus comes from the interpretation of the wavefunctions as probability distribution functions for the posi-

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