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ON ARITHMETIC PROGRESSIONS ON GENUS TWO CURVES

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ABSTRACT. We study arithmetic progression in the x-coordinate of rational points on genus two curves. As we know, there are two models for curve C of genus two: $C : y^2 = f_5(x)$ or $C : y^2 = f_6(x)$, where $f_5,\ f_6 \in \mathbf{Q}[x]$, $\deg f_5 = 5$, $\deg f_6 = 6$ and the polynomials $f_5,\ f_6$ do not have multiple roots. First we prove that there exists an infinite family of curves of the form $y^2 = f(x)$, where $f \in \mathbf{Q}[x]$ and $\deg f = 5$, each containing 11 points in arithmetic progression. We also present an example of $F \in \mathbf{Q}[x]$ with $\deg F = 5$ such that, on the curve $y^2 = F(x)$, 12 points lie in arithmetic progression. Next, we show that there exist infinitely many curves of the form $y^2 = g(x)$ where $g \in \mathbf{Q}[x]$ and $\deg g = 6$, each containing 16 points in arithmetic progression. Moreover, we present two examples of curves in this form with 18 points in arithmetic progression.

1. Introduction. Let $f \in \mathbf{Q}[X]$ be a polynomial without multiple roots, and let us consider the curve $C : y^2 = f(x)$. We say that rational points $P_i = (x_i,\ y_i)$ for $i = 1, 2, \dots, n$ are in *arithmetic progression* on the curve C, if rational numbers x_i are in arithmetic progression for $i = 1, 2, \dots, n$. A positive integer n will be called the *length of arithmetic progression* on the curve C. A natural question arises here: How long can arithmetic progression be on the curve $y^2 = f(x)$ with a fixed degree of f? Throughout the whole paper, by a point we mean a rational one.

In the case of polynomials of degree one, this question is equivalent to the question about the number of squares which form an arithmetic progression.

It is not difficult to show that there exists an infinite family $\mathcal{A}_1$ of polynomials of degree one, with the property that, for each $f \in \mathcal{A}_1$,

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