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ORDERS OF VANISHING OF ZEROS OF CHARACTERISTIC p ZETA FUNCTION

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ABSTRACT. Orders of vanishing of zeros of zeta functions have much arithmetic information encoded in them. For the absolute zeta function, Dinesh Thakur gave sufficient conditions for the order of vanishing of its zeros when the finite field has two elements. Such conditions consider only principal ideals. This result was generalized by Thakur and Diaz-Vargas. Now the conditions involve not only the principal ideals but all the classes of ideals, still in the field of two elements. In this work, we generalize these results to arbitrary finite fields, using similar proofs of Thakur and Diaz-Vargas.

1. Introduction. One of the most important topics in the study of zeta functions is the order of vanishing of its zeros. Some results have been found for the characteristic p zeta function and the "trivial" zeros that we analyze. In [**7**], Thakur gave sufficient conditions for a hyperelliptic function field over the finite field $\mathbf{F}_q$, $q = 2$, to have order of vanishing 2 at the negative integer $-s$. An interesting phenomenon is that such conditions involve the sum of the digits in the expansion base 2 of s, $l_2(s)$, see Theorem 5. This result was generalized by Thakur and Diaz-Vargas in [**2**] considering now all the ideal classes in the definition of the zeta function, and not only the principal ideals. The conditions to have order of vanishing at least 2, depend again on the decomposition base 2 of es, where e is the exponent of the ideal class group, see Theorem 8.

In [**7**], Thakur says succinctly how to deal with the general case $q = p^n$ and arbitrary function fields, when one considers only principal ideals. The conditions in order to have multiplicity q at the zeros, depend now on Weierstrass gaps at ∞. We add an extra condition and give the proof of Theorem 7, which is a generalization of Thakur's theorem for $q = 2$. We analyze also what it means for a function field to have an r-gap structure at ∞.

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