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ORBITS OF LINEAR OPERATORS TENDING TO INFINITY

VLADIMÍR MÜLLER AND JAN VRŠOVSKÝ

ABSTRACT. Let T be a bounded linear operator on a (real or complex) Banach space X satisfying $\sum_{n=1}^{\infty} 1/\|T^n\| < \infty$. Then there is a unit vector $x \in X$ such that $\|T^n x\| \to \infty$. If X is a complex Hilbert space, then it is sufficient to assume that $\sum_{n=1}^{\infty} 1/\|T^n\|^2 < \infty$. The above results are the best possible. We also show analogous results for weak orbits.

1. Introduction. Let X be a real or complex Banach space. Denote by $\mathcal{L}(X)$ the set of all bounded linear operators on X. The *orbit* of a point $x \in X$ under an operator $T \in \mathcal{L}(X)$ is the sequence $(T^n x)_{n=1}^{\infty}$ of vectors. Analogously, the weak orbit of $x \in X$ and $x^* \in X^*$ is the sequence $(\langle T^n x, x^* \rangle)_{n=1}^{\infty}$ of real or complex numbers.

Orbits and weak orbits are closely connected with many fields of operator theory, for example local spectral theory, semi-groups of operators and especially with the invariant subspace/subset problem. For a broader study, see [**3**, Chapter III] or [**5**].

It is still an open problem whether each operator on a Hilbert space (or more generally on a reflexive Banach space) has a nontrivial closed invariant subset (on l_1 a negative solution to this problem was given by Read [**6**]). It is easy to see that an operator T has a nontrivial closed invariant subset if and only if there is a nonzero vector x such that its orbit is not dense.

This paper studies the existence of orbits tending to infinity, i.e., $\|T^n x\| \to \infty$ as $n \to \infty$. This is an easy way to obtain a nondense orbit and therefore a nontrivial closed invariant subset.

By the Banach-Steinhaus theorem, an operator $T \in \mathcal{L}(X)$ has unbounded orbits if and only if $\sup \|T^n\| = \infty$. With orbits tending to

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