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# ON THE DIOPHANTINE EQUATION $(x^2+k)(y^2+k)=(z^2+k)^2$

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ABSTRACT. In this paper we show that if $k \in \mathbf{Z}$ can be represented in the form $k = \pm(a^2 - 2b^2)$, then there exists an infinite family of three-term geometric progressions of numbers of the form $x^2 + k$. Furthermore, we prove that the set of $k \in \mathbf{Z}$, such that there exists a four-term geometric progression of numbers of the form $x^2 + k$ is infinite.

**1. Introduction.** Schinzel and Sierpiński in [**1**] showed that all solutions of the equation

$$(x^2-1)(y^2-1) = \left(\left(\frac{x-y}{2}\right)^2 - 1\right)^2 \tag{1}$$

in positive integers $x$, $y$, $x < y$, are of the form $x = x_n$, $y = x_{n+1}$, $n = 0, 1, \dots$, where $x_0 = 1$, $x_1 = 3$ and generally $x_n = 6x_{n-1} - x_{n-2}$.

Szymiczek in [**2**] generalized the above by showing that all solutions $t$, $x$, $y$, $x < y$ of the equation

$$(x^2-t^2)(y^2-t^2) = \left(\left(\frac{x-y}{2}\right)^2 - t^2\right)^2 \tag{2}$$

in distinct positive integers are of the form

$$t = |m^2 - 2n^2|s, \quad x = (m^2 + 2n^2)s, \quad y = (3m^2 + 8mn + 6n^2)s,$$

where $m$, $n$, $s$ are integers.

Let us point out that (1) and (2) are particular cases of the equation

$$(x^2+k)(y^2+k) = (z^2+k)^2. \tag{3}$$

The question about integer solutions of the equation (3) with fixed $k \in \mathbf{Z}$ is equivalent to the question whether there exists a geometric

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