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GEODESICS AND CURVATURE OF MÖBIUS INVARIANT METRICS

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ABSTRACT. We confirm that certain circular arcs are geodesics for both the Ferrand and Kulkarni-Pinkall metrics. We demonstrate that 'most' Kulkarni-Pinkall isometries are Möbius transformations. We analyze the generalized Gaussian curvatures of these metrics. We exhibit numerous illustrative examples.

1. Introduction. This article is a continuation of [**5, 6**] wherein we studied a Möbius invariant metric $\mu_\Omega(z)|dz|$ introduced by Kulkarni and Pinkhall [**8**] as a canonical metric for Möbius structures on n-dimensional manifolds. In [**5**] we employed the definition given below, see subsection 2.E, and corroborated various properties of this metric using classical function theory. In [**6**] we established pointwise and uniform estimates between the Kulkarni-Pinkall metric and the hyperbolic and quasi-hyperbolic metrics.

Here we examine both the Kulkarni-Pinkall metric and a related metric first studied by Ferrand in [**3**]. We show that certain curves are always geodesics for these metrics, confirm that many Kulkarni-Pinkall isometries are Möbius transformation, and investigate the generalized Gaussian curvatures of both metrics. We also prove a number of basic facts concerning the Kulkarni-Pinkall metric.

Throughout this paper Ω is a region on the Riemann sphere $\widehat{\mathbf{C}}$ with at least two boundary points. Circular geodesics are one of the central objects of our study: we call Γ a *circular geodesic* in Ω if there exists a disk D in $\widehat{\mathbf{C}}$ with $D \subset \Omega$ and such that Γ is a hyperbolic geodesic line in D with endpoints in $\partial D \cap \partial\Omega$. (See below for all definitions.)

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