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ON RECURRENCE RELATIONS FOR THE EXTENSIONS OF EULER SUMS

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ABSTRACT. We consider the extended Euler sums

$$E_{p,q} = \sum_{k=1}^{\infty} \frac{1}{k^q} \left(\sum_{j=1}^{2k} \frac{1}{j^p} \right) \quad \text{and} \quad T_{p,q} = \sum_{k=1}^{\infty} \frac{1}{k^q} \left(\sum_{j=1}^{[k/2]} \frac{1}{j^p} \right)$$

and obtain the explicit values of $E_{p,q}$ and $T_{p,q}$ when the weight $p+q$ is odd via integral transformations of Bernoulli identities involving Bernoulli polynomials. Two families of Bernoulli identities are transformed into explicit formulæ of Euler sums and extended Euler sums.

1. Preliminaries. The sequence of Bernoulli numbers ($n = 0, 1, 2, \dots$) is defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!}, \quad |t| < 2\pi. \tag{1.1}$$

It is a sequence of rational numbers and it can be evaluated through the following recursive formula:

$$\begin{cases} B_0 = 1, \\ \binom{n}{1} B_{n-1} + \cdots + \binom{n}{n} B_0 = 0, n \geq 2. \end{cases} \tag{1.2}$$

In particular, we get $B_1 = -1/2$ from the relation

$$2B_1 + B_0 = 0.$$

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