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IDENTITY-PRESERVING EMBEDDINGS OF COUNTABLE RINGS INTO 2-GENERATOR RINGS

K.C. O'MEARA, C.I. VINSONHALER AND W.J. WICKLESS

ABSTRACT. A technique is presented for embedding countable rings with identity into 2-generator rings with identity so that the embedding respects the identity elements and the centers. As applications we provide a number of examples of finitely generated rings with interesting pathology.

1. Introduction. A common theme in the study of algebraic structures is the embedding of a given structure into a less complicated one. In this note we consider the problem of embedding countable rings into 2-generator rings so that the identity element is preserved. Embeddings not preserving the identity have been constructed by several authors. Each of the papers [**1**, **4** and **5**] presents a method for embedding countable rings into 2-generator rings, but none of the methods respects the identity. We will use a modification of the ideas in [**5**] to solve the more difficult identity-preserving problem. Our embedding has the added advantage of respecting the centers. The payoff is a variety of interesting consequences, some known, but others which we were unable to find in the literature. For example:

(1) Embedding $\mathbf{Q}$ into a 2-generator ring A provides an example of a countable-dimensional $\mathbf{Q}$-algebra A which cannot be decomposed as $A \cong \mathbf{Q} \otimes_{\mathbf{Z}} R$, for R a ring which is free as a $\mathbf{Z}$-module.

(2) A slight modification of the embedding technique permits the construction of a finitely generated primitive ring R with non-zero socle such that eRe is not finitely generated for some primitive idempotent e, and R^*, the group of units of R, is not finitely generated. Thus, although "being finitely generated" is a Morita invariant, associated structures do not, in general, inherit this "finitely generated" property.

(3) Any countable commutative ring can be made the center of a 2-generator ring.

(4) There exists a 2-generator simple ring of characteristic zero.

It is perhaps worth noting that a group-theoretic analogue of (4) is