ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 3, Summer 1989

STABILITY IN WITT RINGS AND ABELIAN SUBGROUPS OF PRO-2-GALOIS GROUPS

ROGER WARE

Let F denote a field of characteristic not two, let $F(2)$ be the maximal 2-extension (i.e., quadratic closure) of F, and let $G_F(2) = \mathrm{Gal}(F(2)/F)$. In this note we introduce and study an invariant, $a(F)$, of F defined to be the largest integer m such that $G_F(2)$ has a closed torsion free abelian subgroup of rank m. [If there is no largest m, we define $a(F) = \infty$]. Recall that the (absolute) *stability index* of F, introduced in [**5**], is $\mathrm{st}\,(F) = \min\{n \mid I^{n+1}F = 2I^nF\}$, where IF is the fundamental ideal of even dimensional forms in the Witt ring, WF, of anisotropic quadratic forms over F and I^nF is the n^{th} power of this ideal. The connection between $a(F)$ and $\mathrm{st}(F)$ will be investigated and it will be shown that if F is a pythagorian field or a finitely generated extension of a hereditarily euclidean or hereditarily quadratically closed base field, then $a(F) = \mathrm{st}\,(F)$. First we present a few examples.

EXAMPLES. (1). $a(F) = 0$ if and only if F is either euclidean (i.e., formally real with $|\dot{F}/\dot{F}^2| = 2$) or quadratically closed if and only if $F(\sqrt{-1})$ is quadratically closed.

(2). If F is a finite, local, or global field, then $a(F) = 1$.

(3). Let F be a rigid field (i.e., every element $a \notin \pm\dot{F}^2$ satisfies $F^2 + aF^2 \subseteq F^2 \cup aF^2$) with $|\dot{F}/\dot{F}^2| = 2^m > 2$. Then $a(F) = m$ if and only if F is nonreal and $F(\sqrt{-1})$ contains all 2-power roots of unity: $a(F) = m - 1$, otherwise. Special cases: $a(\mathbf{R}((t_1))\dots((t_m))) = a(\mathbf{C}((t_1))\dots((t_m)) = a(\mathbf{F}_p((t_1))\dots((t_m))) = m$.

(4). If $F = \mathbf{R}(t_1, \dots, t_n)$ or $\mathbf{C}(t_1, \dots, t_n)$, then $a(F) = n$.

PROOF. (1) is clear and (4) will follow from Theorem 3.

(2). This is clear if F is finite. If F is local or global it can be proved using standard algebraic number theory (cf. [**8**] for a corresponding result for the absolute Galois group). We give an argument that uses