ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 3, Summer 1989

SIMULTANEOUS SIMILARITIES OF PAIRS OF 2×2 INTEGRAL SYMMETRIC MATRICES

OLGA TAUSSKY

This is a continuation of a previous paper [**12**]. The issue of the present paper is stated in a different way as the title indicates. The title links it with S. Friedland's important paper [**2**] called 'Simultaneous similarity of matrices'. However, he deals with matrix pairs with complex entries, even pairs of symmetric ones. My paper is a small inroad in the case of integral matrices.

I will first report briefly on my previous work, including [**12**], and then come to new material. Both parts deal with integral matrices $\mathbf{A}$ with characteristic polynomial $x^2 - m$, $m \equiv 2$, $3(4)$ and square free. The matrix $\mathbf{A} = (a_{ik})$ is 2×2 and belongs to a matrix class in the sense of the theorem of Latimer and MacDuffee.

By a theorem of Frobenius, $\mathbf{A}$ can be expressed as $\mathbf{S}_1\mathbf{S}_2$, with $\mathbf{S}_i$ symmetric and rational. I had studied the problem to characterize the $\mathbf{A}$'s with both factors integral [**10**]. The factorization can be linked to a similarity, say $\mathbf{S}$, between $\mathbf{A}$ and its transpose $\mathbf{A}'$:

$$\mathbf{A}' = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} \quad \text{or} \quad \mathbf{A} = \mathbf{S}\mathbf{A}'\mathbf{S}^{-1}$$

It is known, see, e.g., [**14**] that $\mathbf{S}$ can be chosen symmetric and rational, even integral. In this case also $\mathbf{A}'\mathbf{S}^{-1}$ turns out symmetric and rational. In 1973 I showed that both factors can be chosen integral if and only if the ideal class corresponding to the matrix class of $\mathbf{A}$ is of order 1, 2, 4, apart from a set of m's which will be discussed again in Part II.

Part I. I made an attempt to unify all the m's by expressing $\mathbf{A}$ as the product of two rational matrices $\mathbf{T}_1, \mathbf{T}_2$ with $\mathbf{T}_i = \mathbf{S}^{-1}\mathbf{S}_i\mathbf{S}$ so that $\mathbf{A} = \mathbf{S}^{-1}\mathbf{S}_1\mathbf{S} \cdot \mathbf{S}^{-1}\mathbf{S}_2\mathbf{S}$.

This was done in [**12**] paper in the following way: Instead of studying a single matrix class, all matrix classes corresponding to m are considered, and, in particular, the classes of order 1 or 2 or 4. While there may not be any of order 2 or 4, there is certainly one of order 1, namely