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PROPER EMBEDDING INTO A UNIT LATTICE

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0. Introduction. An n-dimensional quadratic lattice is a free module of rank n over the rational integer ring $\mathbf{Z}$, which is endowed with a symmetric bilinear form B. Let E_n be a unit lattice, that is an n-dimensional quadratic lattice which has an orthonormal basis with respect to B, i.e.,

$$E_n = \mathbf{Z}e_1 + \cdots + \mathbf{Z}e_n, \quad B(e_i, e_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Let A be a positive integer. A sublattice F of E_n is an r-frame of scale A if

$$F = \mathbf{Z}f_1 + \cdots + \mathbf{Z}f_r, \quad B(f_i, f_j) = A\delta_{ij}.$$

A frame F in E_n is proper if $B(F, e_j) \neq \{0\}$ for each j. In this situation we have a problem:

($*$) When does E_n contain a proper r-frame of scale A?

We shall give a complete answer in the case of $r = 2$. Why proper? The Siegel Mass Formula can answer the question: When does E_n contain an r-frame of scale A?

This problem leads to diophantine equations in the following:

(#) E_n contains a proper 1-frame of scale A if and only if there are integers $x_1, \ldots, x_n$ in $\mathbf{Z}$ satisfying

$$x_1^2 + \cdots + x_n^2 = A, \quad x_1 \neq 0, \ldots, x_n \neq 0;$$

(##) E_n contains a proper 2-frame of scale A if and only if there are integers $x_1, \ldots, x_n, y_1, \ldots, y_n$ in $\mathbf{Z}$ satisfying

$$x_1^2 + \cdots + x_n^2 = y_1^2 + \cdots + y_n^2 = A, \quad x_1y_1 + \cdots + x_ny_n = 0,$$
$$x_1 \neq 0 \text{ or } y_1 \neq 0, \ldots, x_n \neq 0 \text{ or } y_n \neq 0.$$

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